On cellular convection driven by surface-tension gradients: effects of mean surface tension and surface viscosity

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The onset of steady, cellular convection driven by surface tension gradients on a thin layer of liquid is examined in an extension of Pearson's (1958) stability analysis. By accounting for the possibility of shape deformations of the free surface it is found that there is no critical Marangoni number for the onset of stationary instability and that the limiting case of 'zero wave-number' is always unstable. Surface viscosity of a Newtonian interface is found to inhibit stationary instability. A simple criterion is found for distinguishing visually the dominant force, buoyancy or surface tension, in cellular convection in liquid pools.

1. Introduction

In small-scale fluid mechanics, of the sorts pertaining to chemical and petroleum engineering and to biology, for example, there is need for knowledge of the roles that interfacial regions between fluid phases may play in driving as well as impeding fluid flow. Mathematical analysis of the onset of hydrodynamic instability driven by surface-tension gradients and influenced by other surface properties is one means for studying these roles. The first such analysis to appear was made by Pearson (1958); it is extended in §§3–5 along lines suggested by our independent analysis of the onset of interfacial-tension-driven convection in superposed fluid layers of large extent (Sternling & Scriven 1959).

Because flows actually powered by surface or interfacial tension have been overlooked or misconstrued so often, there seems to be a need for simple criteria by which they can be recognized. In the case of cellular convection in liquid pools, a means of distinguishing surface-tension-driven from buoyancy-driven convection is required. In §7 it is shown that one is provided by the opposite relations between the directions of surface deflexion and flow produced by the two mechanisms.

It is interesting that the development of the theory of convective instability was promoted by a still-prevalent misinterpretation of Bénard's cells, photographs of which continue to appear as illustrations of natural convection. A resumé of the conflict between predictions from the theory of the onset of natural convection on the one hand, and experimental measurements with shallow liquid pools on the other, will show the physical background for the mathematical analysis that follows.

In his classic experiments on the cellular circulation patterns that occur in a very shallow (0·5–1 mm) pool of liquid heated from below, Bénard (1900, 1901) observed hexagonal cells with a spacing of 3·27 or more times the liquid depth. Bénard's experiments inspired Rayleigh's (1916) analysis of the stability with respect to buoyancy-driven convection of a fluid layer heated beneath. Rayleigh's analysis indicates that *if* hexagonal cells are formed the ratio of their spacing to the layer depth might be about 3·28. The agreement, which impressed Bénard (1927, 1928), is illusory, we know now.

Subsequent extensions of Rayleigh's analysis by Jeffreys, Low and notably Pellew & Southwell (1940) show that the value of the disturbance wave-number at marginal stability, on which the prediction of cell spacing is based, depends appreciably on the boundary conditions at top and bottom of the layer. Rayleigh assumed free surfaces maintained at constant temperature; the others dealt also with rigid boundaries at constant temperature. Jeffreys alone considered boundaries at which heat flux rather than temperature is held fixed. His analysis of this case is incorrect, however, owing to errors in the mathematical statement of the boundary conditions. But Bénard's pool of liquid rested on a rigid plate maintained at substantially constant temperature, and at its free upper surface there was neither constant temperature nor constant heat flux. Rather, in view of temperature distributions found by Bénard, the situation at the upper interface may have corresponded to a constant heat-transfer coefficient. Nor was the surface tension sufficiently high to hold the free surface flat, as tacitly assumed by Rayleigh and others after him. Extensions of Rayleigh's analysis to boundary conditions more closely matching those that actually obtain in experiments with pools of liquid have not yet been published. Thus no prediction of cell spacing in buoyancy-driven convection yet exists with which the observations of Bénard (and others) may properly be compared. And even were such predictions available from calculations of marginal stability, their pertinence to flows sufficiently strong to be observable might be questioned.

According to Rayleigh's analysis, the vertical temperature gradient must attain a certain minimum value for marginal stability and exceed it for instability to occur. Subsequent extensions show that this value too depends significantly on the boundary conditions. Between Rayleigh's case of two free surfaces and Low's of rigid surface below and free surface above, the critical value increases 68 %. Low & Brunt (1925) seem to have been the first to notice that the gradients in Bénard's experiments were at least tenfold *less* than required by Rayleigh's theory; later Bénard himself recognized the discrepancy (1927, 1928) and estimated the ratio at 10^{-4} or 10^{-5} (1930). Vernotte (1936*a*, *b*), recalculated the data and put the ratio at roughly 10^{-2}. It is unlikely that so large a discrepancy arises solely in inaccurate boundary conditions.

Bénard observed upwelling of hot liquid always below the centres of *depression* of the free surface of the pool. He and his successors (e.g. Volkovisky 1939) reported this to be the case in unsteady as well as steady cellular convection, even with the pool flowing as a film down an inclined surface. Here is the first unquestionable conflict with the theory: noting that in buoyancy-driven flows the free surface over an upwelling current is generally *elevated*, Jeffreys (1951) proceeded to show that the theory begun by Rayleigh indeed requires this behaviour. It is interesting that Bénard, although he intended to avoid attempting to explain the cellular patterns, recognized the conflict in his original papers and suggested that surface tension is somehow responsible.

To resolve the surface-deflexion anomaly Jeffreys, evidently unaware that Volkovisky and others had already done so, recommended that Bénard's experimental work be repeated. Block's (1956) brief report tells of more than mere repetition. He found again that cellular convection can occur when the temperature gradient is at least an order of magnitude smaller than required by the theory of Rayleigh, Jeffreys, and Low. And whereas the theory predicts stability of a layer *cooled* from below, Block observed Bénard cells in a shallow pool so cooled! Moreover, Block discovered that putting a film of silicone—an insoluble surface-active agent of very low surface tension—on a shallow pool heated beneath brings the convection to a half, but that adding liquid to increase the depth of the arrested pool causes circulation to resume when the combination of depth and temperature gradient becomes about that required by the theory. He also found that traces of contamination, which are difficult to avoid (Hickman 1952), on a water surface have an inhibitory effect. Block drew the conclusion that Bénard cells in shallow pools are produced by variations in surface tension, which are in turn due to non-uniformities of temperature over the free surface (non-uniformities of composition can produce the same effect). This mechanism, sometimes called the 'Marangoni effect', manifests itself in a variety of phenomena (Scriven & Sternling 1960). Block implied that it would account for the surface depressions over upwelling hot liquid.

Drying paint films often display Bénard cells, and when they do the circulation is observed whether the free surface is made the underside or the topside of the paint layer. The circulation therefore cannot be caused by the buoyancy mechanism. This fact led to Pearson's (1958) theoretical demonstration, by means of a small disturbance analysis and independently of Block's work, that surface tension forces suffice to cause hydrodynamic instability in a liquid layer with a free surface, provided there is a temperature or concentration gradient of proper sense and sufficient magnitude across the layer. Pearson's theory agrees in many essentials with the experimental findings he and Block have reported, and together they have illuminated a neglected type of surface-tension-driven flow. Nevertheless, the picture is far from complete in general outline, much less in detail.

There is a question as to whether the effectively infinite surface tension tacitly assumed in Pearson's analysis confers greater stability on the model he analysed than exists in reality. In other words, what roles do flexibility and resistance to deformation of the surface play in determining stability? Pearson's theory predicts a critical depth, usually rather less than that given by the Rayleigh–Jeffreys–Low theory, below which there is stability relative to convection induced by surface tension. Although Pearson cites experiments with evaporating films which seem to substantiate this prediction. Block observed Bénard cells in exceedingly shallow pools and found no indication of a critical depth; but unfortunately he reported insufficient data to permit a quantitative comparison. In the same connexion the inhibitory action of relatively insoluble surface-active materials, referred to by both Block and Pearson, deserves analysis in the light of the departure from equilibrium tension that is produced by deformation of an interfacial film (cf. Scriven & Sternling 1960). There is another matter of fundamental significance. According to Pearson's theory, which considers only the stationary régime of neutral stability—i.e. 'marginal convective instability'—the pool if unstable with a temperature or concentration gradient of one sense must be stable with a gradient of the opposite sense. However, Block seems to have observed Bénard cells in comparable pools, one heated and the other cooled below, although the patterns may not have been as regular in the latter case. For certain unbounded systems of contiguous fluid phases in which convection is induced by interfacial tension, stationary regimes have been predicted for gradients of one sense and a combination of oscillatory régimes (i.e. 'overstability') with unusual stationary régimes for gradients of the opposite sense (Sternling & Scriven 1959). This suggests that an extension of Pearson's analysis to include the possibility of oscillatory régimes might reveal similar behaviour by a liquid layer subject to surface tension forces, perhaps accounting for Block's observation. The possibility of overstability by the buoyancy mechanism was disproved by Pellew & Southwell (1940) but remains an open question so far as the surface-tension mechanism is concerned. Bénard reported permanently unsteady cellular convection in shallow pools of highly volatile liquids. Volkovisky mentioned the appearance of 'turbulent' flow in pools more than 3–4 mm deep.

By extending Pearson's small-disturbance analysis to a still idealized yet more realistic model of the fluid interface, we are able to establish the effects of finite mean surface tension and of surface viscosity. The model used is that of a Newtonian fluid interface, in which the local departure from equilibrium interfacial stress is directly proportional to the local rate of interfacial strain, the proportionality constants being independent of direction in the interface. In the next section the dynamical equations for such an interface are simplified for a surface that departs only infinitesimally from a plane. These equations are then the crucial boundary conditions to be applied at the free surface of a liquid layer subjected to infinitesimal disturbances.

In § 3 the instability problem is formulated and carried to the point of a formal solution, equations (33), from which the development in time of any elementary, infinitesimal disturbance can be computed. In § 4 it is shown that disturbances having a vorticity component perpendicular to the layer all are damped, and so there is indeed justification for omitting them from consideration. The subsequent section contains results of calculations of the conditions necessary to the existence of neutrally stable disturbances of the steady, or stationary, kind—the marginal convective instability mentioned above. Calculations for oscillatory instability, or overstability, are sufficiently more complex that we do not undertake here to search for the oscillatory régimes whose possible existence was suggested above. Section 6 contains remarks on the role of free interfaces in hydrodynamic instability and the criterion for distinguishing driving mechanisms is presented in the last section.

The method used below to reduce the equations to dimensionless form may be of minor interest. It differs from the conventional one by obviating the need to construct a definite unit of measurement for a physical variable when no natural unit occurs in the problem statement, as for velocity here.

2. Dynamics of a Newtonian fluid interface

We shall be concerned with a surface infinitesimally deformed from a plane, which we take as the original and mean position of the free surface of the liquid layer. For simplicity all lengths are measured in units of mean layer depth d. In Cartesian co-ordinates (x, y, z) the perturbed surface is given by

$$z = 1 + (B^*/d)\, B(x,y).$$

B^* denotes the maximum deflexion of the perturbed surface, the unit in which local surface deflexion is to be measured; symbols bearing asterisks are used throughout to denote units of measurement. To the first order in the relative amplitude of the deflexion, the unit normal $\mathbf{n}$ to the perturbed surface points in the z-direction, i.e. $\mathbf{n} = \mathbf{k}$, and the mean curvature H is given by

$$2H = \left(\frac{B^*}{d}\right) \nabla_{II}^2 B, \quad \nabla_{II}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

To the same approximation the surface gradient operator in the interface is equivalent to the two-component gradient operator for surfaces of constant z

$$\nabla_{II} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}.$$

The kinematic condition on the bulk-phase velocity $\mathbf{V}$ at a free interface of negligible, or at least sensibly constant mass, is

$$\mathbf{k}.\mathbf{V} = (B^*/V^*\tau^*)\, \partial B/\partial \tau. \tag{1}$$

where τ is time; and, according to the equations of motion of a Newtonian fluid interface (Scriven 1960; cf. Aris 1962), the dynamic condition for an interface of negligible mass is

$$-\mathbf{F} = (\delta\sigma^*/F^*d)\, \nabla_{II}\sigma + \{(\kappa+\epsilon)\, V^*/F^*d^2\}\, \nabla_{II}(\nabla_{II}.\mathbf{V}) + (\epsilon V^*/F^*d^2)\, \mathbf{k}\times\nabla_{II}(\mathbf{k}.\nabla\times\mathbf{V}) + (\sigma^*/F^*d)\, 2H\sigma\mathbf{k}, \tag{2}$$

to the first order in (B^*/d) again. Here σ is the (dimensionless) mean, equilibrium, surface tension; κ and ϵ are, respectively, the surface dilational and shear viscosities, which are assumed to be uniform; and $\delta\sigma^*$, σ^*, V^*, and F^* are as yet unspecified units of surface-tension variation, surface tension, velocity, and force, respectively. The viscous traction exerted on the interface by gas standing overhead is ordinarily negligible. The traction $\mathbf{F}$ exerted on the interface by the incompressible liquid beneath is as usual given by

$$-\mathbf{F} = -(P^*/F^*)\,\mathbf{k}P + (\mu V^*/F^*d)\,\mathbf{k}.[\nabla\mathbf{V} + (\nabla\mathbf{V})^\dagger], \tag{3}$$

where P is the excess of the local thermostatic pressure over the ambient pressure and $\nabla\mathbf{V} + (\nabla\mathbf{V})^\dagger$ is the local rate of strain. Equating the z-components of (2) and (3), we have the balance of normal forces at the interface

$$(d\mu V^*/B^*\sigma^*)\,[-(P^*d/\mu V^*)\,P + 2DV] = \sigma\nabla^2_{II}\,B,\ddagger \tag{4}$$

where $$V \equiv \mathbf{k}.\mathbf{V} \quad \text{and} \quad D \equiv \partial/\partial z.$$

Equating the surface divergences of (2) and (3), we obtain from the tangential force balance and the continuity equation a second condition on V at the interface

$$-D^2V + \nabla^2_{II}\,V = (\delta\sigma^*/\mu V^*)\,\nabla^2_{II}\,\sigma - (\kappa + \epsilon/\mu d)\,\nabla^2_{II}(DV). \tag{5}$$

Finally, by equating the surface curls of (2) and (3), we extract a third independent scalar relation from the original vector equation; this one, heretofore overlooked, is a condition on the z-component of vorticity at the interface

$$DW = (\epsilon/\mu d)\,\nabla^2_{II}\,W, \tag{6}$$

where $$W \equiv \mathbf{k}.\nabla\times\mathbf{V} \equiv \mathbf{k}.\nabla_{II}\times\mathbf{V}.$$

Equations (4), (5) and (6) are free-surface boundary conditions on flow in a liquid layer subjected to infinitesimal disturbances.

3. Mathematical formulation

Following Pearson, but with somewhat different notation, we take for the undisturbed, steady-state system a quiescent liquid layer whose surface at $z = 0$ lies against a solid body, whose free surface at $z = 1$ is in contact with an inviscid fluid, and whose temperature is a linear function of the z co-ordinate alone. Next, we superpose an infinitesimal disturbance and linearize the equations of motion and heat transport. In their general dimensionless form with body forces omitted§ these equations, which apply in a liquid layer of any

‡ If the effect of gravity at the perturbed interface were to be included, the term $-(\rho g d^2/\sigma^*)\;B\sin\theta$ would be added to the right-hand side of this equation, θ being the angle the force of gravity makes with the mean position of the free surface.

§ Since the liquid is supposed to be incompressible, the potential of any conservative force field such as gravity can be accommodated, if necessary, in the pressure term. However, it would be necessary to include gravity explicitly in equation (8) were it desired to account for density variations producing appreciable buoyancy forces, as in a study of coupling with buoyancy-induced instability (or stability).

It would be necessary to include gravity in the boundary condition for the normal component of traction at equation (26) were it desired to focus on the peculiarities of disturbances having gravity-wave in contrast to capillary-wave character, i.e. wavelength λ such that the familiar ratio of gravity to capillary forces in a wavy surface,

$$(\rho g\lambda^2/4\pi^2\sigma^*)\sin\theta = (\rho g d^2/\sigma^*)\sin\theta/\alpha^2,$$

is not small. Such would be the case in a study of coupling with Rayleigh–Taylor instability or the converse situation of stability.

orientation, are

$$\nabla . \mathbf{V} = 0; \tag{7}$$

$$\left(\frac{d^2}{\nu\tau^*}\right)\frac{\partial \mathbf{V}}{\partial \tau} = -\left(\frac{P^* d}{\mu V^*}\right)\nabla P + \nabla^2 \mathbf{V}; \tag{8}$$

$$\left(\frac{d^2}{\mathscr{D}\tau^*}\right)\frac{\partial T}{\partial \tau} + \left(\frac{\mathscr{L} V^* d^2}{\mathscr{D} T^*}\right)\mathbf{k}.\mathbf{V} = \nabla^2 T. \tag{9}$$

Here ν is the kinematic and μ the dynamic viscosity; $\mathscr{D}$ is the thermal diffusivity; $\mathscr{L}$ is the undisturbed temperature at $z = 1$ less that at $z = 0$, divided by the layer thickness d; τ^* is a characteristic time in units of which time τ is to be measured; T^* is a characteristic temperature disturbance; and so forth. With the aid of standard vector operators and identities we obtain from (7) to (9) a set of scalar equations:

$$\nabla^2 P = 0, \tag{10}$$

$$\left[\left(\frac{d^2}{\nu\tau^*}\right)\frac{\partial}{\partial \tau} - \nabla^2\right]\nabla^2 V = 0, \tag{11}$$

$$\left[\left(\frac{d^2}{\nu\tau^*}\right)\frac{\partial}{\partial \tau} - \nabla^2\right] W = 0, \tag{12}$$

$$\left[\left(\frac{d^2}{\mathscr{D}\tau^*}\right)\frac{\partial}{\partial \tau} - \nabla^2\right] T = -\left(\frac{\mathscr{L} V^* d^2}{\mathscr{D} T^*}\right) V. \tag{13}$$

As the only direction of physical distinction is that denoted by $\mathbf{k}$, and $V \equiv \mathbf{k}.\mathbf{V}$ and $W \equiv \mathbf{k}.\nabla \times \mathbf{V}$ lie along it, they are conveniently referred to as longitudinal components of velocity and vorticity, respectively. Equations (10) to (12) for pressure and these two variables are quite independent. The longitudinal component of velocity can be eliminated from equation (13) to get an equation in temperature alone:

$$\left[\left(\frac{d^2}{\mathscr{D}\tau^*}\right)\frac{\partial}{\partial \tau} - \nabla^2\right]\left[\left(\frac{d^2}{\nu\tau^*}\right)\frac{\partial}{\partial \tau} - \nabla^2\right]\nabla^2 T = 0. \tag{14}$$

To simplify these equations we assume that the disturbance behaves exponentially with time, that its longitudinal (z) and transverse (x, y) dependencies are separable, and that its transverse structure is periodic and wave-like in the sense of satisfying the Helmholtz equation in the plane (cf. Pellew & Southwell 1940). That is, we assume:

$$P = e^{\beta\tau} f(x,y)\, p(z), \quad V = e^{\beta\tau} f(x,y)\, v(z), \quad W = e^{\beta\tau} f(x,y)\, w(z),$$
$$T = e^{\beta\tau} f(x,y)\, t(z), \quad B = e^{\beta\tau} f(x,y)\, b_0,$$

with $\nabla^2_{II} f = -\alpha^2 f$. Here β is the dimensionless time constant, in general complex. The separation constant α defines the size (though not the shape) of cells in the periodic transverse structure; it is the dimensionless 'wave-number', 2π times the ratio of layer thickness to a mean size, or wavelength, of a cellular pattern. Equation (14) becomes

$$(D^2 - \alpha^2 q^2)(D^2 - \alpha^2 r^2)(D^2 - \alpha^2)\, t = 0, \tag{15}$$

where $\quad q^2 \equiv 1 + \mathscr{N}_{Pr}\xi, \quad r^2 \equiv 1 + \xi, \quad \mathscr{N}_{Pr} \equiv \nu/\mathscr{D}, \quad \xi \equiv \beta d^2/\alpha^2 \nu\tau^*.$

Equations (10) to (13) become

$$(D^2-\alpha^2)p = 0, \tag{16}$$

$$[(D^2-\alpha^2r^2)(D^2-\alpha^2)]v = 0, \tag{17}$$

$$(D^2-\alpha^2r^2)w = 0, \tag{18}$$

$$(D^2-\alpha^2q^2)t = (\mathscr{L}V^*d^2/\mathscr{D}T^*)v. \tag{19}$$

A useful relation between pressure and the longitudinal component of velocity follows from (7), (8) and (16)

$$(P^*d/\mu V^*)p = \alpha^{-2}(D^3-\alpha^2r^2D)v. \tag{20}$$

Thus once the solution of equation (15) for the longitudinal distribution of temperature is found, the longitudinal velocity distribution is given by differentiation and (19), while the longitudinal distribution of pressure is given by further differentiation and (20). In an analysis by the method of small disturbances it is not necessary to have the functional form of the transverse structure, $f(x,y)$, which defines shape in a cellular pattern; it would be required, however, were the vector velocity field of a disturbance desired.

The transverse components of velocity are represented by

$$\mathbf{V}_{II} = \mathbf{V}-\mathbf{k}V.$$

It can be shown, e.g. from the familiar identity

$$\nabla^2\mathbf{V} = -\nabla\times\nabla\times\mathbf{V}+\nabla(\nabla.\mathbf{V}),$$

that in general (cf. Sani & Scriven 1964)

$$-\nabla^2_{II}\mathbf{V}_{II} = \mathbf{V}_{II}DV+\mathbf{k}D(\nabla.\mathbf{V})-\mathbf{k}\times\nabla W,$$

from which it follows in the present case that

$$\mathbf{V}_{II} = \alpha^{-2}[(DV)\nabla_{II}f-w\mathbf{k}\times\nabla_{II}f].$$

Consequently, were we to specify the planform of a disturbance the solutions of (15) and (18) would provide a complete history of perturbations of the sort contemplated until such time as non-linear effects become important.

Six boundary conditions on velocity and temperature are needed for the solution of (15). Three of these occur at the solid surface at $z = 0$; the other three occur at the perturbed free surface $z = 1+B^*B/d$ but are confounded with the deflexion B, which, being caused by the disturbance, is an additional unknown. For this reason a fourth condition at the free surface is needed, making a total of seven boundary conditions rather than six.

The boundary conditions at $z = 0$ are:

$$V = 0, \quad \text{whence by (19)} \quad (D^2-\alpha^2q^2)t = 0, \tag{21}$$

$$\mathbf{V}_{II} = 0, \quad \text{whence} \quad DV = 0 \quad \text{and} \quad (D^3-\alpha^2q^2D)t = 0, \tag{22}$$

and either

$$T = 0, \quad \text{whence} \quad t = 0, \tag{23a}$$

for Pearson's 'conducting' case—constant temperature at $z = 0$—or

$$DT = 0, \quad \text{whence} \quad Dt = 0, \tag{23b}$$

for his 'insulating' case—constant heat flux at $z = 0$. These reduce the general solution of (15) to forms involving only three arbitrary constants:

$$\begin{aligned} t = {} & A_1[\mathcal{N}_{Pr}(\sinh \alpha r z - r \sinh \alpha z) + r \sinh \alpha z] \\ & + A_2[\mathcal{N}_{Pr}(\cosh \alpha r z - \cosh \alpha z) - (\cosh a q z - \cosh a z)] \\ & + A_3 \sinh \alpha q z \end{aligned} \tag{24a}$$

for the conducting case; and

$$\begin{aligned} t = {} & A_1[\mathcal{N}_{Pr}(\cosh \alpha r z - \cosh \alpha z) + \cosh \alpha z] \\ & + A_2[\mathcal{N}_{Pr}(\sinh \alpha r z - r \sinh \alpha z) - r(q^{-1} \sinh \alpha q z - \sinh \alpha z)] \\ & + A_3 \cosh \alpha q z \end{aligned} \tag{24b}$$

for the insulating case.

Although the remaining four conditions apply at the perturbed free surface, they may be rewritten as boundary conditions at $z = 1$. If we retain the leading terms of the Taylor expansion of temperature about $z = 1$ we have

$$T = T(1) + (\mathcal{L}B^*/T^*)\,B.$$

If we assume as usual that surface tension is a linear function of temperature of the deflected surface we have

$$\sigma = \sigma_0 + (\zeta T^*/\sigma^*)\,[T(1) + (\mathcal{L}B^*/T^*)\,B]$$

and $$\nabla_{II}^2 \sigma = (\zeta T^*/\delta\sigma^*)\,[\nabla_{II}^2 T(1) + (\mathcal{L}B^*/T^*)\,\nabla_{II}^2 B],$$

where ζ is the differential coefficient of surface-tension change with temperature. Then, to the first order in perturbations and regardless of the units in which velocity and pressure perturbations are measured, the kinematic condition (1) becomes

$$\alpha^{-2}(D^2 - \alpha^2 q^2)\,t - (\mathcal{L}B^*/T^*)\,\mathcal{N}_{Pr}\,\xi b_0 = 0; \tag{25}$$

the normal force balance (4) becomes

$$(\mu\mathcal{D}/\sigma^* d)\,\alpha^{-4}[(D^2 - \alpha^2 r^2 - 2\alpha^2)(D^2 - \alpha^2 q^2)\,D]\,t - (\mathcal{L}B^*/T^*)\,b_0 \sigma_0 = 0; \tag{26}$$

and the tangential force balance (5) reduces to

$$\frac{1}{\alpha^2}[(D^2 + \alpha^2)(D^2 - \alpha^2 q^2)]\,t + \left(\frac{\kappa + \epsilon}{\mu d}\right)[(D^2 - \alpha^2 q^2)\,D]\,t - \left(\frac{\zeta\mathcal{L}d^2}{\mu\mathcal{D}}\right)t - \left(\frac{\zeta\mathcal{L}d^2}{\mu\mathcal{D}}\right)\left(\frac{\mathcal{L}B^*}{T^*}\right)b_0 = 0. \tag{27}$$

The fourth condition is the requirement that energy be conserved, which to the first order demands continuity of heat flux at the free surface, and in that way leads to

$$Dt + (\mathcal{H}d/\rho c\,\mathcal{D})\,t + (\mathcal{H}d/\rho c\,\mathcal{D})\,(\mathcal{L}B^*/T^*)\,b_0 = 0, \tag{28}$$

where $\mathcal{H}$ is the heat-transfer coefficient, assumed constant.

In these equations B^* denotes the maximum deflexion of the perturbed surface. The units σ^* and T^* in which surface-tension and temperature perturbations, respectively, are to be measured remain at our disposal. We choose σ^* to be the undisturbed surface tension so that $\sigma_0 = 1$, and T^* such that $(\mathcal{L}B^*/T^*) = 1$. Defining several dimensionless parameters:

Crispation group,	$\mathcal{N}_{Cr} \equiv \mu\mathcal{D}/\sigma^* d;$
Surface viscosity group,	$\mathcal{N}_{Vi} \equiv (\kappa + \epsilon)/\mu d;$
Marangoni number,	$\mathcal{N}_{Ma} \equiv \zeta\mathcal{L}d^2/\mu\mathcal{D};$
Nusselt number,	$\mathcal{N}_{Nu} \equiv \mathcal{H}d/\rho c\mathcal{D};$

we then have at $z = 1$:

$$(D^2 - \alpha^2 q^2)\, t - \mathscr{N}_{Pr}\,\xi b_0 = 0, \tag{29}$$

$$\mathscr{N}_{Cr}[(D^2 - \alpha^2 r^2 - 2\alpha^2)(D^2 - \alpha^2 q^2)\, D]\, t - \alpha^4 b_0 = 0, \tag{30}$$

$$[(D^2 + \alpha^2)(D^2 - \alpha^2 q^2)]\, t + \alpha^2 \mathscr{N}_{Vi}[(D^2 - \alpha^2 q^2)\, D]\, t - \alpha^2 \mathscr{N}_{Ma}\, t - \alpha^2 \mathscr{N}_{Ma}\, b_0 = 0, \tag{31}$$

$$Dt + \mathscr{N}_{Nu}\, t + \mathscr{N}_{Nu}\, b_0 = 0. \tag{32}$$

This set of equations suffices to determine, though not uniquely, the three remaining arbitrary constants and the unknown surface deflexion, provided the wave number α, the time constant β (now disguised as ξ), and the dimensionless parameters have such values that a non-trivial solution to the set exists. For a solution to exist the determinant of the matrix of coefficients of A_1, A_2, A_3 and b_0 after equation (24) is substituted in (29) to (32) must vanish; this in turn requires that the following characteristic equation be satisfied:

$$\frac{\mathscr{N}_{Ma}}{\alpha\mathscr{N}_{Pr}(1-\mathscr{N}_{Pr})\,\xi} = \frac{\begin{vmatrix} S_r - rS & C_r - C & 0 & -1 \\ r[2(C_r - C) - \xi C] & 2(rS_r - S) - \xi S & 0 & 1/\alpha\mathscr{N}_{Cr}\mathscr{N}_{Pr}\xi \\ \begin{array}{l} 2(S_r - rS) + \xi S_r \\ \quad + \alpha\mathscr{N}_{Vi}\, r(C_r - C) \end{array} & \begin{array}{l} 2(C_r - C) + \xi C_r \\ \quad + \alpha\mathscr{N}_{Vi}(rS_r - S) \end{array} & 0 & 0 \\ \begin{array}{l} \alpha\mathscr{N}_{Pr}\, r(C_r - C) \\ \quad + \mathscr{N}_{Pr}\mathscr{N}_{Nu}(S_r - rS) \\ \quad + r(\alpha C + \mathscr{N}_{Nu} S) \end{array} & \begin{array}{l} \alpha\mathscr{N}_{Pr}(rS_r - S) \\ \quad + \mathscr{N}_{Pr}\mathscr{N}_{Nu}(C_r - C) \\ \quad + \alpha(S - qS_q) + \mathscr{N}_{Nu}(C - C_q) \end{array} & \begin{array}{l} \alpha q C_q \\ \quad + \mathscr{N}_{Nu} S_q \end{array} & \mathscr{N}_{Nu}(1 - \mathscr{N}_{Pr}) \end{vmatrix}}{\begin{vmatrix} S_r - rS & C_r - C & 0 & -1 \\ r[2(C_r - C) - \xi C] & 2(rS_r - S) - \xi S & 0 & 1/\alpha\mathscr{N}_{Cr}\mathscr{N}_{Pr}\xi \\ \mathscr{N}_{Pr}(S_r - rS) + rS & \mathscr{N}_{Pr}(C_r - C) + (C - C_q) & S_q & 1 - \mathscr{N}_{Pr} \\ \mathscr{N}_{Pr}\, r(C_r - C) + rC & \mathscr{N}_{Pr}(rS_r - S) + S - qS_q & qC_q & 0 \end{vmatrix}} \tag{33a}$$

for the conducting case; and

$$\frac{\mathscr{N}_{Ma}}{\alpha\mathscr{N}_{Pr}(1-\mathscr{N}_{Pr})\,\xi} = \frac{\begin{vmatrix} C_r - C & S_r - rS & 0 & -1 \\ 2(rS_r - S) - \xi S & 2r(C_r - C) - \xi rC & 0 & 1/\alpha\mathscr{N}_{Cr}\mathscr{N}_{Pr}\xi \\ \begin{array}{l} 2(C_r - C) + \xi C_r \\ \quad + \alpha\mathscr{N}_{Vi}(rS_r - S) \end{array} & \begin{array}{l} 2(S_r - rS) + \xi S_r \\ \quad + \alpha\mathscr{N}_{Vi}\, r(C_r - C) \end{array} & 0 & 0 \\ \begin{array}{l} \alpha\mathscr{N}_{Pr}(rS_r - S) \\ \quad + \mathscr{N}_{Pr}\mathscr{N}_{Nu}(C_r - C) \\ \quad + \alpha S + \mathscr{N}_{Nu} C \end{array} & \begin{array}{l} \alpha\mathscr{N}_{Pr}\, r(C_r - C) \\ \quad + \mathscr{N}_{Pr}\mathscr{N}_{Nu}(S_r - rS) \\ \quad + \alpha r(C - C_q) - \mathscr{N}_{Nu}\, r(S_q - qS)/q \end{array} & \begin{array}{l} \alpha q S_q \\ \quad + \mathscr{N}_{Nu} C_q \end{array} & \mathscr{N}_{Nu}(1 - \mathscr{N}_{Pr}) \end{vmatrix}}{\begin{vmatrix} C_r - C & S_r - rS & 0 & -1 \\ 2(rS_r - S) - \xi S & 2r(C_r - C) - \xi rC & 0 & 1/\alpha\mathscr{N}_{Pr}\mathscr{N}_{Pr}\xi \\ \mathscr{N}_{Pr}(C_r - C) + C & \mathscr{N}_{Pr}(S_r - rS) + r(S_q - qS)/q & C_q & 1 - \mathscr{N}_{Pr} \\ \mathscr{N}_{Pr}(rS_r - S) + S & \mathscr{N}_{Pr}\, r(C_r - C) + r(C - C_q) & qS_q & 0 \end{vmatrix}} \tag{33b}$$

for the insulating case. Here

$$S \equiv \sinh\alpha, \quad S_q \equiv \sinh\alpha q, \quad S_r \equiv \sinh\alpha r,$$
$$C \equiv \cosh\alpha, \quad C_q \equiv \cosh\alpha q, \quad C_r \equiv \cosh\alpha r,$$

and the original determinant in each case has been partitioned to isolate the Marangoni number as a factor. These equations are the formal solution of the entire instability problem, for they permit the time course of any type of infinitesimal disturbance to be computed once its wave-number, α, and the various dimensionless parameters of the system are specified.

4. Longitudinal vorticity

For the solution of (18) in the longitudinal (z) component of vorticity two boundary conditions are needed. They are $\mathbf{V} = 0$ at $z = 0$, whence $\nabla_{II} \times \mathbf{V} = 0$ there; and (6) at $z = 1$. Thus

$$(D^2 - \alpha^2 r^2)\, w = 0, \tag{18}$$

$$w = 0 \quad \text{at} \quad z = 0, \tag{34}$$

$$Dw + \alpha^2(\epsilon/\mu d)\, w = 0 \quad \text{at} \quad x = 1. \tag{35}$$

Clearly these are satisfied by any disturbance having transverse structure such that longitudinal vorticity is totally absent. Transverse structures with longitudinal vorticity are also possible, but are restricted to a discrete set of wave-numbers; namely, those satisfying the ancillary characteristic equation,

$$\alpha r \coth \alpha r = -\alpha^2(\epsilon/\mu d), \tag{36}$$

obtained by substituting the solution of (18) and (34) in (35).

In stability considerations, however, there is no need to go beyond the trivial solution, $w = 0$: any disturbance possessing longitudinal vorticity is damped and therefore cannot lead to instability. In mathematical terms, multiplication of (18) by the complex conjugate $\tilde{w}$, integration over the layer depth, and introduction of (34) and (35) through integration by parts leads to

$$-\alpha^2\left(\frac{\epsilon}{\mu d}\right)\tilde{w}(1)\, w(1) - \int_0^1 D\tilde{w}\, Dw\, dz - \alpha^2 r^2 \int_0^1 \tilde{w} w\, dz = 0. \tag{37}$$

It follows immediately, since $\alpha^2(r^2 - 1) = \beta(d^2/\nu\tau^*)$, that

$$\beta = -\left(\frac{\nu\tau^*}{d^2}\right)\left[\int_0^1 (D\tilde{w}\, Dw + \alpha^2 \tilde{w} w)\, dz + \left(\frac{\epsilon}{\mu d}\right)\alpha^2 \tilde{w}(1)\, w(1)\right] \Big/ \int_0^1 \tilde{w} w\, dz. \tag{38}$$

Because the right-hand side is negative definite the time factor is real and negative. In physical terms there is no source and, in creeping flow, no production of longitudinal vorticity in the liquid layer; moreover, the solid surface is a passive boundary and the free surface a dissipative boundary with respect to longitudinal vorticity. Thus this vorticity component can only diminish in magnitude thereafter if at any instant it is not totally absent. We therefore consider it no further in this analysis.

5. Neutrally stable stationary disturbances

The liquid layer may be subject to two kinds of instability, according as the exponential time constant β, now disguised as ξ, is real or complex. The first is the stationary régime (often called 'convective instability') in which the disturbance grows steadily in place; the second is the oscillatory régime ('overstability') in which the growing disturbance displays temporal periodicity. In either case the real part of the time constant must be positive, for if it is negative the disturbance dies out in time. Thus we can solve for the conditions under which the layer first becomes unstable, i.e. neutral (or 'marginal') stability by setting the real part of β equal to zero in the characteristic equation. If in so doing we set the imaginary part also equal to zero, we can find only stationary régimes—fully time-independent motions. This is what Pearson did, and what we shall do in this section, deferring discussion of the possibility of equally significant oscillatory régimes—time-dependent neutrally stable motions.

If we set $\beta = 0$ we have $\xi = 0$, $q = r = 1$, and the solutions (24) satisfying the boundary conditions at $z = 0$ become

$$t = A_1' z(az \cosh \alpha z - \sinh \alpha z) + A_2' z(\alpha z \sinh \alpha z - 3 \cosh \alpha z) + A_3' \sinh \alpha z, \quad (39a)$$

$$t = A_1' z(\alpha z \cosh \alpha z - \sinh \alpha z) + A_2'(\alpha^2 z^2 \sinh \alpha z - 3\alpha z \cosh \alpha z + 3 \sinh \alpha z) + A_3' \cosh \alpha z. \quad (39b)$$

The characteristic equation reduces to

$$\mathcal{N}_{Ma} = \frac{8\alpha(\alpha \cosh \alpha + \mathcal{N}_{Nu} \sinh \alpha)\,[\alpha - \sinh \alpha \cosh \alpha + (\mathcal{N}_{Vi}/2)\,\alpha(\alpha^2 - \sinh^2 \alpha)]}{\alpha^3 \cosh \alpha - \sinh^3 \alpha - 8\,\mathcal{N}_{Cr} \alpha^3 \cosh \alpha} \quad (40a)$$

for the conducting case, and

$$\mathcal{N}_{Ma} = \frac{8\alpha(\alpha \sinh \alpha + \mathcal{N}_{Nu} \cosh \alpha)\,[\alpha - \sinh \alpha \cosh \alpha + (\mathcal{N}_{Vi}/2)\,\alpha(\alpha^2 - \sinh^2 \alpha)]}{\alpha^3 \sinh \alpha - \alpha^2 \cosh \alpha + 2\alpha \sinh \alpha - \sin^2 \alpha \cosh \alpha - 8\,\mathcal{N}_{Cr} \alpha^3 \sinh \alpha} \quad (40b)$$

for the insulating case.

In the limit of vanishing surface viscosity and infinitely large surface tension $\mathcal{N}_{Vi} = \mathcal{N}_{Cr} = 0$ and these forms simplify to the characteristic equations obtained earlier by Pearson. In the absence of surface viscosity or an equivalent dissipative surface effect the interfacial engine itself would convert surface energy to kinetic energy reversibly regardless of the viscosity of the adjacent liquid. With infinite tension the free surface would preserve, as may be seen from equation (26), an unnatural flatness.

Equations (40) are graphed in figures 1 to 6, the curves there representing neutral stability with respect to stationary disturbances. Each curve separates a region of stable conditions to the left from one of the unstable conditions to the right of it. The extensions of the curves beyond the confines of the figures are evident from the limiting behaviour of the Marangoni number, shown in table 1. The most striking aspect of these curves is their dependence on the physics of the boundaries, particularly the thermal behaviour of both boundary surfaces and the elastic behaviour of the free one. In the conducting case the liquid layer is always unstable with respect to disturbances of small wave-number (great wave-length) and there is, strictly speaking, no *critical* Marangoni number (except in the mathematical limit of $\mathscr{N}_{Cr} = 0$). That is, there is no value of the Marangoni number below which disturbances of all wave-numbers are damped. However, as $\mathscr{N}_{Cr}$ diminishes (increasing surface tension) the spectrum of wave-numbers is barely at first, then ever more sharply divided into two ranges of instability

	Conducting case	Insulating case	
		$\mathscr{N}_{Nu} = 0$	$\mathscr{N}_{Nu} \neq 0$
$\alpha \to 0$, $\mathscr{N}_{Cr} = 0$	$80/\alpha^2$	48	$48\mathscr{N}_{Nu}/\alpha^2$
$\mathscr{N}_{Cr} \neq 0$	$2\alpha^2/3\mathscr{N}_{Cr}$	$2\alpha^2/3\mathscr{N}_{Cr}$	$2\mathscr{N}_{Nu}/3\mathscr{N}_{Cr}$
$\alpha \to \infty$, $\mathscr{N}_{Vi} = 0$	$8\alpha^2$	$8\alpha^2$	
$\mathscr{N}_{Vi} \neq 0$	$4\alpha^3\mathscr{N}_{Vi}$	$4\alpha^3\mathscr{N}_{Vi}$	

TABLE 1. Limiting behaviour of Marangoni number for neutral stability

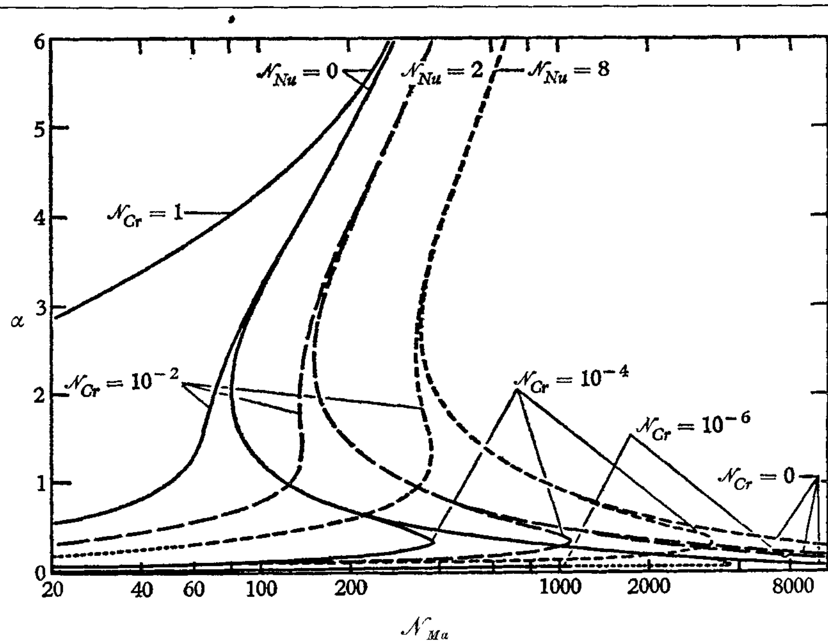

FIGURE 1. Neutral stability curves, conducting case ($\mathscr{N}_{Vi} = 0$).

as shown in figure 1. In the insulating case the situation is the same when the heat flux at the free surface is also maintained constant ($\mathscr{N}_{Nu} = 0$); otherwise there does exist a critical Marangoni number, as shown in figure 2. In both cases it is clear that the mathematical limit of $\mathscr{N}_{Cr} = 0$ is unrealistic at small wave-numbers, although it is an accurate approximation at higher wave-numbers provided the crispation group, $\mathscr{N}_{Cr}$, is not much more than 10^{-4}. This group varies widely in magnitude, for example, from 2×10^{-6} for a water layer of 1 mm thickness to a value thousands of times greater for a 100μ layer of viscous organic liquid. In the original experiments by Bénard it was probably of the order of

10^{-5} or 10^{-4}. For the reinvestigation by Volkovisky (1939) more definite estimates may be possible for some of the liquids he used.

In circumstances such that all wave-numbers are unstable or two distinct ranges of wave-numbers are unstable even more caution than usual must be exercised in drawing inferences about physical reality from the linearized

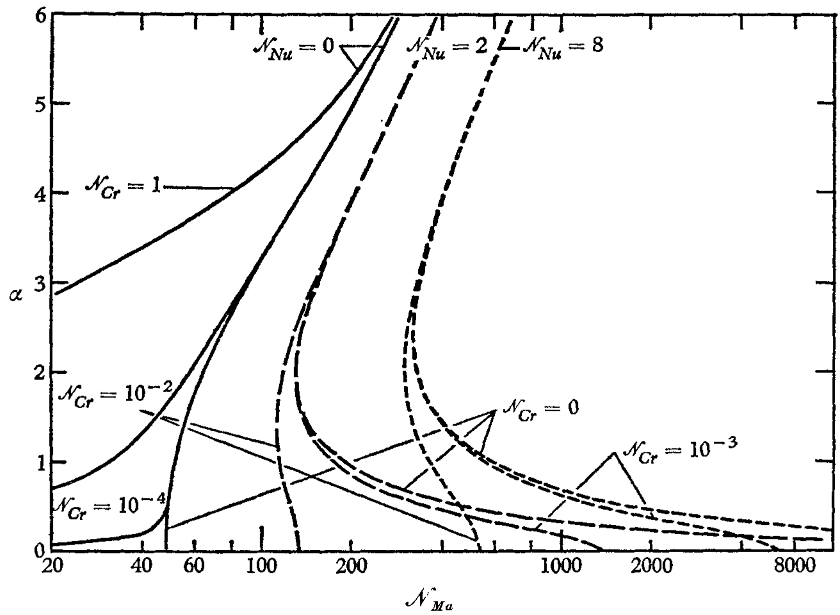

FIGURE 2. Neutral stability curves, insulating case ($\mathcal{N}_{Vi} = 0$).

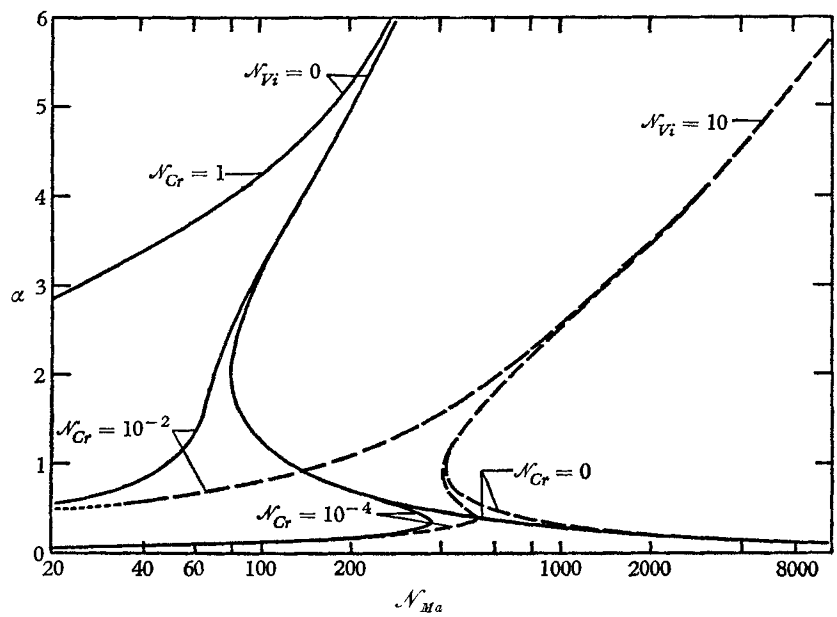

FIGURE 3. Effect of dissipation in the interface, conducting case ($\mathcal{N}_{Nu} = 0$).

stability analysis. Whatever the grounds in simpler circumstances for predicting properties of a resultant fully developed flow from a related neutral stability calculation, the basis of prediction shifts to the more involved computation of fastest-growing wave-numbers, i.e. the wave number for which the real part of the

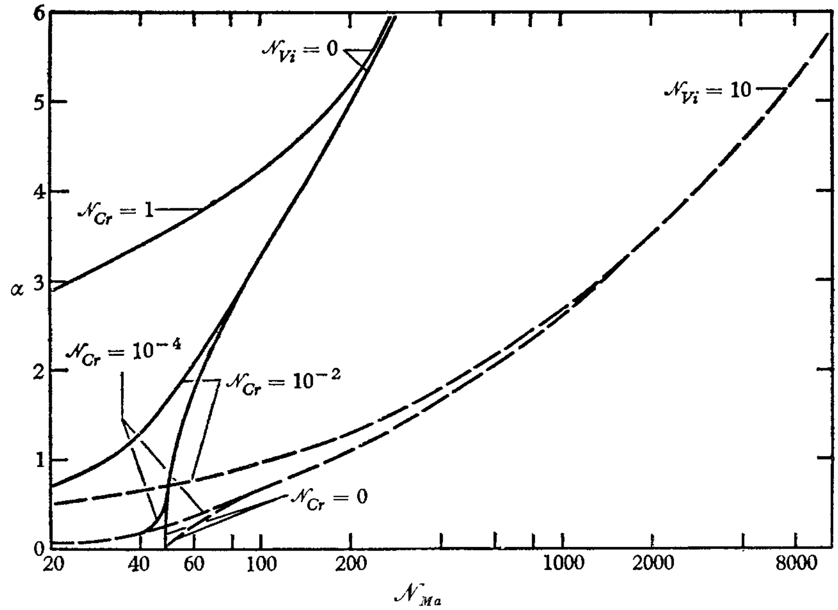

FIGURE 4. Effect of dissipation in the interface, insulating case ($\mathcal{N}_{Nu} = 0$).

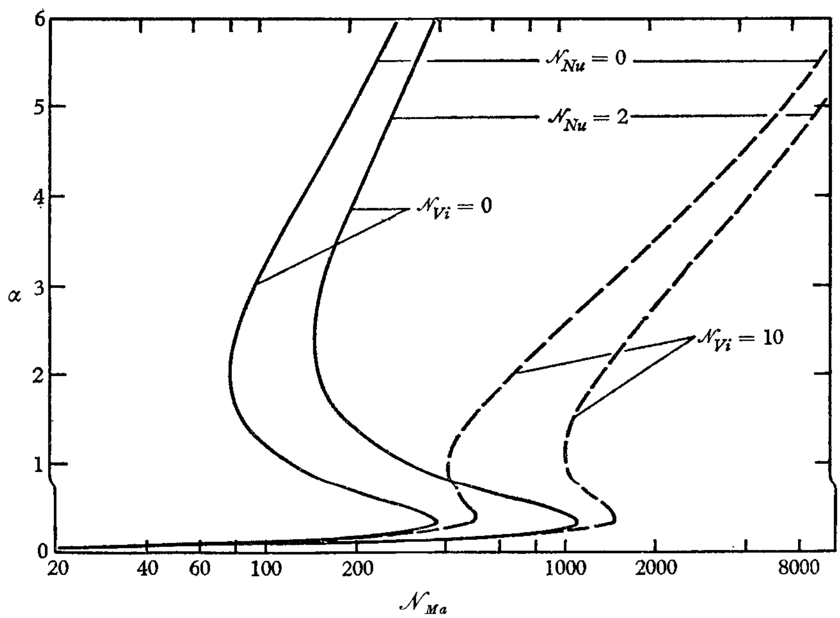

FIGURE 5. Interaction of dissipation and heat transfer at the interface, conducting case ($\mathcal{N}_{Cr} = 10^{-4}$).

exponential time factor is greatest. Until the computation is made we can only surmise that curves of constant growth rate parallel the neutral stability curve except at the smallest and very largest wave-numbers which if they grow at all do so very slowly indeed (cf. Sternling & Scriven 1959). Thus with no surface viscosity, as in figures 1 and 2, and relatively high surface tension ($\mathcal{N}_{Cr} < 10^{-2}$) the dimensionless wave-number that is usually dominant is probably close to $\alpha = 2$, because there are pronounced local minima in the neutral stability curves in the vicinity of that value. In many other instances shown in figures 1 to 6, particularly those of relatively low surface tension or high surface viscosity,

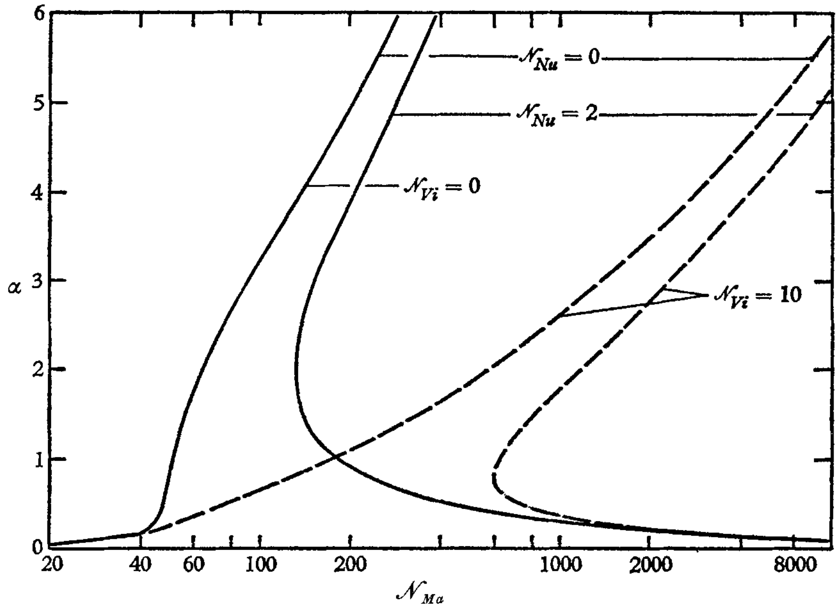

FIGURE 6. Interaction of dissipation and heat transfer at interface, insulating case ($\mathcal{N}_{Cr} = 10^{-4}$).

there is no local minimum to seize upon: further discussion requires consideration of growth rates and possible wave-numbers of disturbances in relation to the maximum spatial dimension and time interval available in the specific physical situation.‡ It may ultimately prove possible to define a quasi-critical Marangoni number in the manner of Brooke Benjamin's (1957) quasi-critical Reynolds number for film flow.

The chief effect of a surface-excess of mechanical energy dissipation brought about by surface viscosity ($\mathcal{N}_{Vi} \neq 0$) is, unsurprisingly, to render the liquid layer more stable or more nearly stable, especially with respect to disturbances of

‡ The effect of gravity requires consideration too, because through the normal stress balance at the interface it can strongly influence flow disturbances for which α is sufficiently small, corresponding to wavelength λ large compared to layer depth. Dr Brooke Benjamin has pointed out in a private communication that for ordinary liquids in horizontal layers 1 mm or more deep the action of gravity becomes significant for wavelengths exceeding about 5 mm and could very well stabilize disturbances of longer wavelength if the free interface is at the top of the layer.

large wave-number (short wavelength), as figures 3 to 6 indicate. One consequence is displacement of local minima in the neutral stability curves to Marangoni numbers five times as great and wave-numbers half as large when the surface viscosity group increases from zero to 10. This group too varies widely in magnitude. For a water layer 1 mm thick it might range from 10^{-3} for a relatively clean interface to 10 or more for one covered by an insoluble monolayer (for a discussion of surface viscosity values see references cited by Sternling & Scriven 1959, Scriven & Sternling 1960).

The influence of boundary conditions is further complicated by the interaction of thermal and mechanical conditions at the free surface, examples of which are provided by figures 5 and 6. Furthermore, the radical differences between the extremes of constant temperature—the 'conducting' case—and constant heat flux at the rigid boundary—the 'insulating' case—indicate a stronger dependence on the effective Nusselt number in the general thermal boundary condition there than was supposed in Pearson's discussion (of his equation (9)).

Thus the story of convection cells induced by surface tension still is far from complete. Each of the preceding four paragraphs provides a basis for questioning Pearson's comparison of his analysis with observation and experiment. However, his conclusion that what Bénard observed was actually surface-tension-driven flow, can be established by consideration of the relation between directions of surface deflexion and of flow, as is done below.

6. Remarks

To disregard the possibility that the free surface is deformed by flow amounts to constraining it to remain perfectly flat, as though by a rigid boundary that somehow allows tangential slip—or by an exceedingly large surface tension. The result, it is now clear, is to confer on the liquid layer greater stability at large wavelengths than exists when the interface deforms elastically. The lower the mean surface tension, the less the stability; thus surface tension has a stabilizing tendency, here with respect to disturbances that are standing waves in essence. Brooke Benjamin (1957) has found that the tendency of surface tension is the same in the formation of travelling waves in film flow. Indeed, it should be expected to have the same effect on any given mode of motion in liquid with a free boundary.

An example is the system of two unequilibrated, immiscible fluids in which convection is induced by interfacial tension, which we have analysed on the simplifying assumption of a permanently flat interface (Sternling & Scriven 1959). It is likely that the constraint implied by this assumption has a stabilizing effect that is not always realistic, just as above.

Another example is the pool of liquid in which convection really is induced by buoyancy, in which case Low's theory for one rigid and one 'free' boundary might be pertinent. But the 'free' boundary of Rayleigh, Jeffreys, and Low, since it too is free only with respect to lateral motion along it, is just as likely to have a stabilizing effect on buoyancy-driven flow in a shallow pool. The mercury pools used by Nakagawa (1957 *a, b*, 1959) in his important experiments are an interesting case. Had the free surface of the mercury been clean and *in vacuo* the crispation group would have been extremely small,

$$\mathcal{N}_{Cr} \sim 1{\cdot}7 \times 10^{-6} d^{-1} \, (d \geqslant 3\,\text{cm});$$

as it was, the value surely was no more than one order-of-magnitude greater: still sufficiently small that the destabilization associated with finite, as opposed to infinite surface tension, was negligible.

Because the boundary conditions corresponding to an inflexible but laterally free surface are simple to handle, particularly in certain variational methods of solving stability problems, they have frequently been imposed. Heretofore the alternative has been to adopt the sometimes less tractable conditions corresponding to a completely rigid boundary. Consideration of the role of surface flexibility in determining stability makes clear that these models of an interface are merely two of many that are latent in linear, homogeneous vector boundary conditions on velocity and traction. Some elementary model interfaces that neither supply nor dissipate mechanical energy are the (i) completely rigid; (ii) laterally rigid but flexible, like a sheet of paper afloat; (iii) inflexible but laterally free as we have discussed; and (iv) completely free, i.e. deformable in both the normal and lateral directions. The listing is in order of decreasing constraint on motions in the vicinity of the interface.

The elementary models become more complicated when they are modified to account for dissipation in the surface, as by the working of linear surface viscosities. It is well-known that surface-active agents suppress wave formation in film flow and inhibit surface-tension-driven flows in general, and it has been suggested that these effects be analysed in terms of surface viscosity. The suggestion can be extended to Nakagawa's experiments with pools of mercury, for he found that surface contamination, which is notoriously difficult to avoid on mercury, inhibits convective instability induced by buoyancy. He indicated that without extensive precautions a film of contamination builds up which is laterally rigid but flexible, or even completely rigid, which seems rather less likely.

The situation becomes still more complicated when, as in convection induced by surface tension, the interface must be represented as the prime mover, the seat of instability. All of these complications can, however, be conveniently modelled with the Newtonian fluid interface, which we have used here, and its generalizations. This approach centring on the fluid-fluid interface should be compared with that taken by Brooke Benjamin (1960) by way of the 'compliant boundary', which is more closely related to conventional idealizations of fluid-solid interfaces.

The important roles of surface tension and surface viscosity in the neutral stability problem studied here point to the need for careful consideration and accurate models of interfacial behaviour in small-scale fluid mechanics. The smaller the scale, the greater in general is the relative importance of interface over bulk, and the more acute is the need.

7. A simple criterion of driving mechanism

The relation between directions of surface deflexion and of flow is established as follows. In a steady motion the longitudinal (z) component of velocity vanishes at the free surface of the liquid, according to the kinematic condition (1). Therefore, if DV is negative at the free surface, the liquid immediately beneath is moving towards the surface. If at the same point the surface deflexion is also negative, then there is upwelling beneath the depression in the surface. Hence the sign of the dimensionless ratio B/DV reveals the desired relation, and does so for the entire surface at once, because of the periodic transverse structure. From equations (4) and (13), or (30) and (19), we get for steady flow ($\xi = \beta = 0$; also choose $V^* = \mathscr{D}T^*/\mathscr{L}d^2$ for convenience)

$$\frac{B}{DV} = \frac{b_0}{Dv} = \frac{\mathscr{N}_{Cr}}{\alpha^4} \frac{[(D^2-3\alpha^2)(D^2-\alpha^2 D)]\,t}{(D^3-\alpha^2 D)\,t} \quad \text{at} \quad z = 1. \tag{41}$$

With (39) we find that in both the insulating and conducting cases.

$$\frac{B}{DV} = \frac{2\mathscr{N}_{Cr}}{\alpha^2} \frac{(A_3'/A_2')(\cosh\alpha - \alpha\sinh\alpha) - \alpha\cosh\alpha}{(A_3'/A_2')\,\alpha\sinh\alpha + \cosh\alpha + \sinh\alpha}. \tag{42}$$

Finally, by invoking the kinematic boundary condition (29) the ratio of constants can be eliminated in both cases to give

$$B/DV = 2\mathscr{N}_{Cr}/\{(\sinh\alpha)^2 - \alpha^2\}. \tag{43}$$

Because $(\sinh\alpha)^2 \geqslant \alpha^2$ and $\mathscr{N}_{Cr}$ is necessarily positive, the right-hand side is always positive. *Therefore in steady cellular convection driven by surface tension, there is upflow beneath depressions and downflow beneath elevations of the free surface*; more accurately, flow is toward the free surface in shallow sections, away in deeper sections. The relationship is just the converse in buoyancy-driven flows, as Jeffreys (1951) showed. In the first situation liquid wells up to fill and thereby reduce the curvature of the depression left as the top layers are swept aside by surface flow; in the second, liquid is driven upward by buoyancy. This contrast is a simple means of distinguishing which of the two mechanisms is chiefly responsible for an observed flow. Thus from Bénard's words and illustrations it is plain that the steady flows he saw were all driven by surface tension. The same is true of those Volkovisky saw in pools of water, ethanol, and various oils. One can now identify other experimenters who have studied surface-tension-driven convection without fully realizing it—from Varley and Weber in the 1850's, who were probably the first to describe cellular convections (see Scriven & Sternling 1960), to Levengood (1959), who described curious secondary features of 'localized thermal instability' in very shallow pools of methanol some 100 years later.

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